# Op-Amp Sallen-Key Filter Design

Eric Danson

September 28, 2024

The relative sensitivity of $y$ to $x$ is defined as

$$S_x^y = \frac{\partial y / y}{\partial x / x} = \left( \frac{\partial y}{\partial x} \right) \frac{x}{y} \tag{1}$$

and is useful for optimizing component values in filter designs. Designing for low sensitivity is a way to minimize variations in filter parameters due to component tolerances.

## 1 Low-Pass Filter

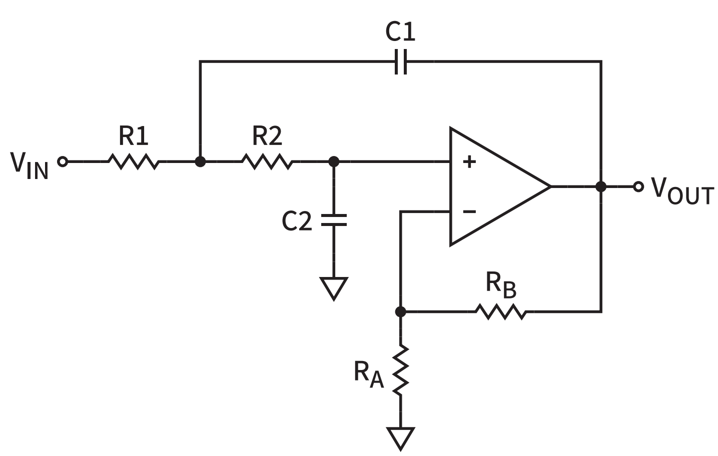


Figure 1: Sallen-Key low-pass filter schematic.

$$H(s) = H_0 \frac{1}{s^2/{\omega_0}^2 + s/(\omega_0 Q) + 1} \tag{2}$$

$$H_0 = K = 1 + R_\mathrm{B}/R_\mathrm{A} \tag{3}$$

$$\omega_0 = \frac{1}{\sqrt{R_1 R_2 C_1 C_2}} \tag{4}$$

$$Q = \frac{\sqrt{R_1 R_2 C_1 C_2}}{(1-K)\, R_1 C_1 + (R_1 + R_2)\, C_2} \tag{5}$$

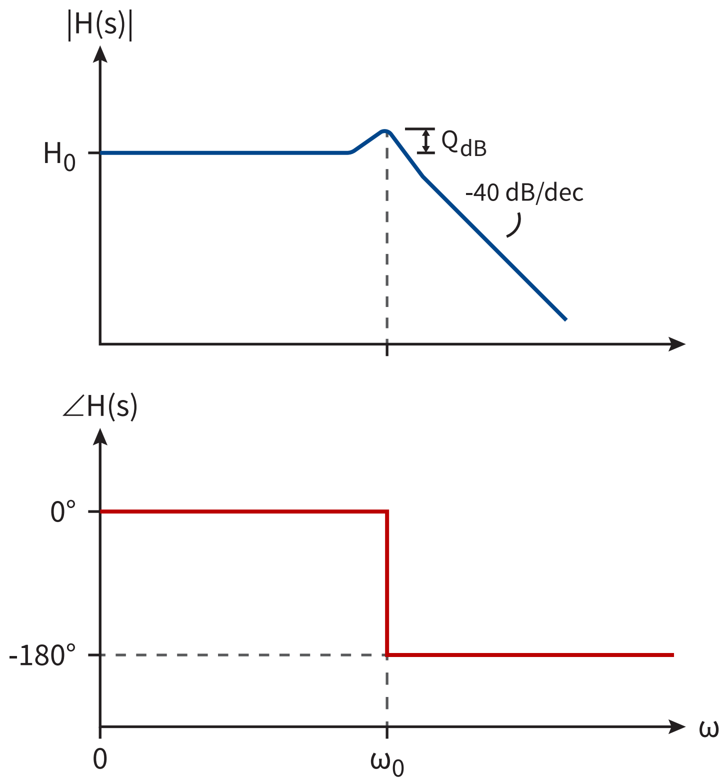


Figure 2: Sallen-Key low-pass filter Bode plot.

## 1.1 Sensitivity

$$S_{R_1}^{\omega_0} = S_{R_2}^{\omega_0} = S_{C_1}^{\omega_0} = S_{C_2}^{\omega_0} = -\frac{1}{2} \tag{6}$$

$$S_{R_B}^{\omega_0} = S_{R_A}^{\omega_0} = 0 \tag{7}$$

$$S_{R_1}^{Q} = -S_{R_2}^{Q} = \omega_0 Q R_2 C_2 - \frac{1}{2} \tag{8}$$

$$S_{C_1}^{Q} = -S_{C_2}^{Q} = \omega_0 Q \left(R_1 C_2 + R_2 C_2\right) - \frac{1}{2} \tag{9}$$

$$S_{R_B}^{Q} = -S_{R_A}^{Q} = \omega_0 Q \left(K - 1\right) R_1 C_1 \tag{10}$$

## 1.2 Components as Ratios

$$\begin{aligned} R_1 &= mR \\ R_2 &= R \\ C_1 &= nC \\ C_2 &= C \end{aligned} \tag{11}$$

$$\omega_0 = \frac{1}{RC\sqrt{mn}} \tag{12}$$

$$Q = \frac{\sqrt{mn}}{\left(1 - K\right) mn + m + 1} \tag{13}$$

### 1.2.1 Minimizing Resistor Sensitivity

$S_{R_1}^{Q} = S_{R_2}^{Q} = 0$ for

$$\omega_0 Q R_2 C_2 = \frac{1}{\left(1 - K\right) mn + m + 1} = \frac{1}{2} \tag{14}$$

$$\implies n = \frac{m - 1}{\left(K - 1\right) m} \tag{15}$$

Substituting $n$ into $Q$ and solving for $m$,

$$m = 1 + 4Q^2 \left(K - 1\right) \tag{16}$$

$$\implies n = \frac{4Q^2}{1 + 4Q^2 \left(K - 1\right)} \tag{17}$$

If $C$ is chosen, then $R$ can be calculated from $\omega_0$ as

$$R = \frac{1}{2\omega_0 QC} \tag{18}$$

### 1.2.2 Minimizing Capacitor Sensitivity

$S_{C_1}^{Q} = S_{C_2}^{Q} = 0$ for

$$\omega_0 Q \left(R_1 C_2 + R_2 C_2\right) = \frac{m+1}{(1-K)\,mn + m + 1} = \frac{1}{2} \tag{19}$$

However, the ratio expression has a minimum value of 1, so the minimum capacity sensitivity is $S_{C_1}^{Q} = -S_{C_2}^{Q} = 1/2$ and occurs when $K = 1$.

### 1.2.3 Monte Carlo Simulation

The circuit used to generate the plot was designed for $K = 1$, $\omega_0 = 2\pi \cdot 1000$, and $Q = 1.5$. The resistor and capacitor tolerances were set to 1% and 5%, respectively.

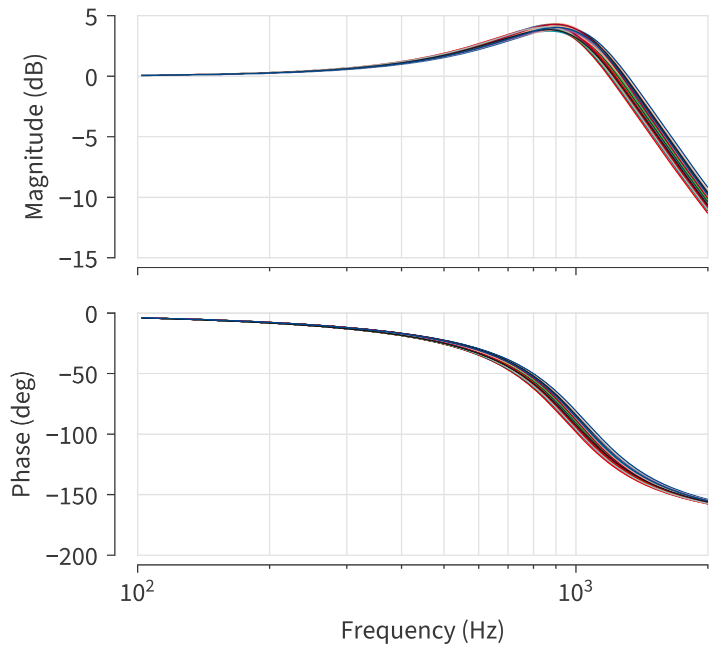


Figure 3: Sallen-Key low-pass filter Bode plot of 100 Monte Carlo samples.

### 1.3 Resistors as Ratios and Equal Capacitors

$$\begin{aligned} R_1 &= mR \\ R_2 &= R \\ C_1 = C_2 &= C \end{aligned} \tag{20}$$

$$\omega_0 = \frac{1}{RC\sqrt{m}} \tag{21}$$

$$Q = \frac{\sqrt{m}}{(2-K)\,m + 1} \tag{22}$$

If $C$ is chosen, then

$$m = \frac{4Q^2}{\left[1 + \sqrt{1 + 4Q^2(K-2)}\right]^2} \tag{23}$$

$$R = \frac{1 + \sqrt{1 + 4Q^2(K-2)}}{2\omega_0 QC} \tag{24}$$

### 1.4 Equal Components

$$\begin{aligned} R_1 &= R_2 = R \\ C_1 &= C_2 = C \end{aligned} \tag{25}$$

$$\omega_0 = \frac{1}{RC} \tag{26}$$

$$Q = \frac{1}{3-K} \tag{27}$$

## 2 High-Pass Filter

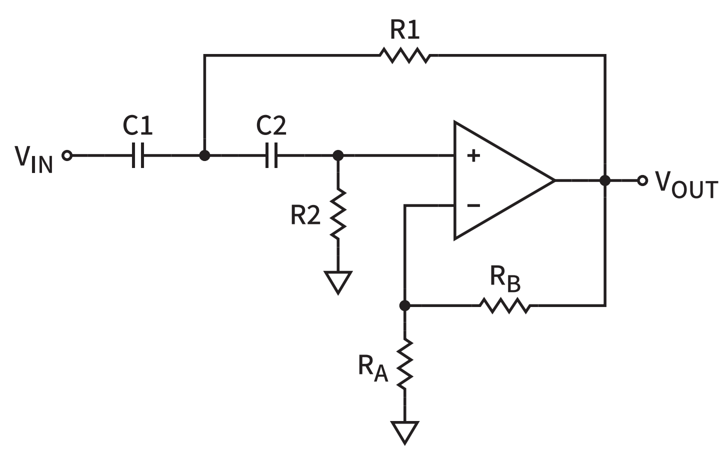


Figure 4: Sallen-Key high-pass filter schematic.

$$H(s) = H_0 \frac{s^2/\omega_0{}^2}{s^2/\omega_0{}^2 + s/(\omega_0 Q) + 1} \tag{28}$$

$$H_0 = K = 1 + R_B/R_A \tag{29}$$

$$\omega_0 = \frac{1}{\sqrt{R_1 R_2 C_1 C_2}} \tag{30}$$

$$Q = \frac{\sqrt{R_1 R_2 C_1 C_2}}{(1-K)\, R_2 C_2 + R_1\,(C_1 + C_2)} \tag{31}$$

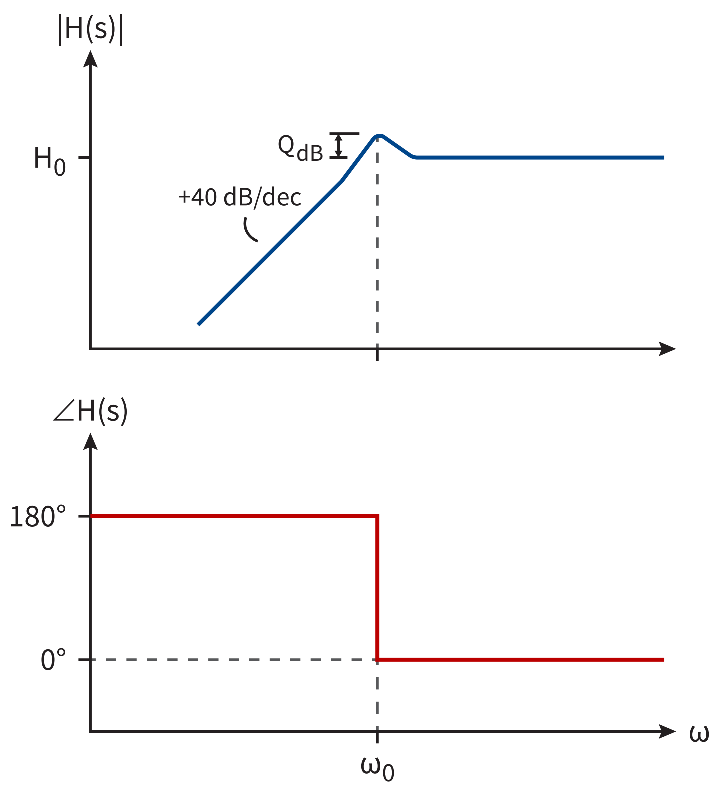


Figure 5: Sallen-Key high-pass filter Bode plot.

## 2.1 Sensitivity

$$S_{R_1}^{\omega_0} = S_{R_2}^{\omega_0} = S_{C_1}^{\omega_0} = S_{C_2}^{\omega_0} = -\frac{1}{2} \tag{32}$$
$$S_{R_\mathrm{B}}^{\omega_0} = S_{R_\mathrm{A}}^{\omega_0} = 0 \tag{33}$$
$$S_{R_2}^{Q} = -S_{R_1}^{Q} = \omega_0 Q \left(R_1 C_1 + R_1 C_2\right) - \frac{1}{2} \tag{34}$$
$$S_{C_2}^{Q} = -S_{C_1}^{Q} = \omega_0 Q R_1 C_1 - \frac{1}{2} \tag{35}$$
$$S_{R_\mathrm{B}}^{Q} = -S_{R_\mathrm{A}}^{Q} = \omega_0 Q \left(K - 1\right) R_2 C_2 \tag{36}$$

## 2.2 Components as Ratios

$$\begin{aligned} R_1 &= mR \\ R_2 &= R \\ C_1 &= nC \\ C_2 &= C \end{aligned} \tag{37}$$

$$\omega_0 = \frac{1}{RC\sqrt{mn}} \tag{38}$$
$$Q = \frac{\sqrt{mn}}{(1 - K) + m\,(1 + n)} \tag{39}$$

### 2.2.1 Minimizing Capacitor Sensitivity

$S_{C_2}^{Q} = S_{C_1}^{Q} = 0$ for

$$\omega_0 Q R_1 C_1 = \frac{mn}{(1 - K) + m\,(1 + n)} = \frac{1}{2} \tag{40}$$
$$\implies n = \frac{(1 - K) + m}{m} \tag{41}$$

Substituting $n$ into $Q$ and solving for $m$,

$$m = \frac{1}{4Q^2} + (K - 1) \tag{42}$$
$$\implies n = \frac{1}{1 + 4Q^2\,(K - 1)} \tag{43}$$

If $C$ is chosen, then $R$ can be calculated from $\omega_0$ as

$$R = \frac{2Q}{\omega_0 C} \tag{44}$$

### 2.2.2 Minimizing Resistor Sensitivity

$S^Q_{R_2} = S^Q_{R_1} = 0$ for

$$\omega_0 Q \left(R_1 C_1 + R_1 C_2\right) = \frac{m\,(1+n)}{(1-K) + m\,(1+n)} = \frac{1}{2} \tag{45}$$

However, the ratio expression has a minimum value of 1, so the minimum resistor sensitivity is $S^Q_{R_2} = -S^Q_{R_1} = 1/2$ and occurs when $K = 1$.

### 2.2.3 Monte Carlo Simulation

The circuit used to generate the plot was designed for $K = 1$, $\omega_0 = 2\pi \cdot 1000$, and $Q = 1.5$. The resistor and capacitor tolerances were set to 1% and 5%, respectively.

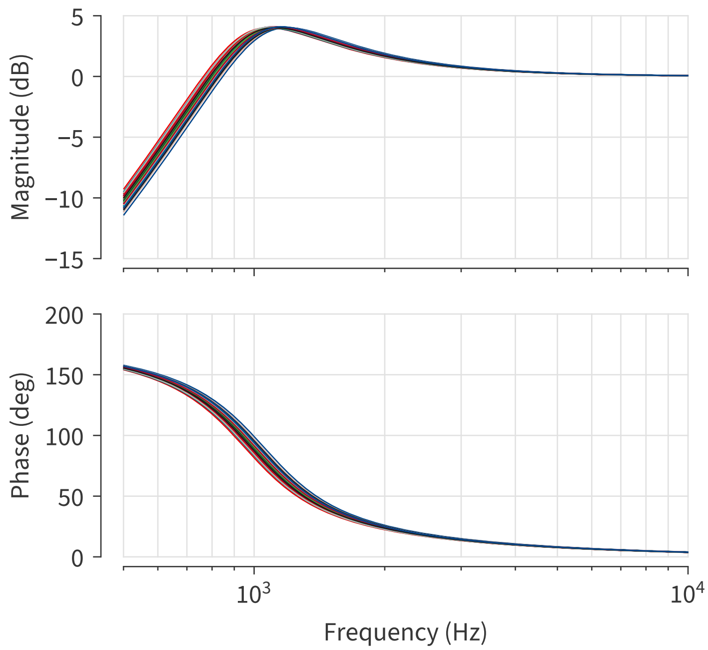


Figure 6: Sallen-Key high-pass filter Bode plot of 100 Monte Carlo samples.

## 2.3 Resistors as Ratios and Equal Capacitors

$$\begin{aligned} R_1 &= mR \\ R_2 &= R \\ C_1 = C_2 &= C \end{aligned} \tag{46}$$

$$\omega_0 = \frac{1}{RC\sqrt{m}} \tag{47}$$

$$Q = \frac{\sqrt{m}}{(1-K) + 2m} \tag{48}$$

If $C$ is chosen, then

$$m = \frac{\left[1 + \sqrt{1 + 8Q^2 (K - 1)}\right]^2}{16Q^2} \tag{49}$$

$$R = \frac{4Q}{\left[1 + \sqrt{1 + 8Q^2 (K - 1)}\right] \omega_0 C} \tag{50}$$

### 2.4 Equal Components

$$\begin{aligned} R_1 &= R_2 = R \\ C_1 &= C_2 = C \end{aligned} \tag{51}$$

$$\omega_0 = \frac{1}{RC} \tag{52}$$

$$Q = \frac{1}{3 - K} \tag{53}$$

## 3 Band-Pass Filter

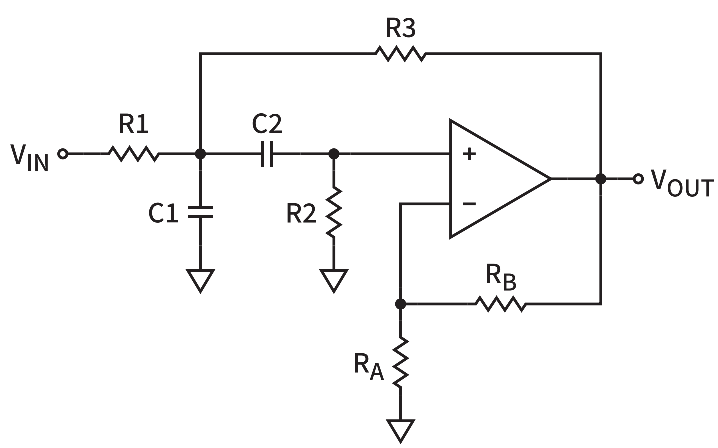


Figure 7: Sallen-Key band-pass filter schematic.

$$H(s) = H_0 \frac{s/(\omega_0 Q)}{s^2/\omega_0{}^2 + s/(\omega_0 Q) + 1} \tag{54}$$

$$K = 1 + R_B/R_A \tag{55}$$

$$H_0 = \frac{K}{1 + (1-K)\,R_1/R_3 + (R_1/R_2)\,(1 + C_1/C_2)} \tag{56}$$

$$\omega_0 = \frac{\sqrt{1 + R_1/R_3}}{\sqrt{R_1 R_2 C_1 C_2}} \tag{57}$$

$$Q = \frac{\sqrt{1 + R_1/R_3} \cdot \sqrt{R_1 R_2 C_1 C_2}}{[1 + (1-K)\,R_1/R_3]\,R_2 C_2 + R_1\,(C_1 + C_2)} \tag{58}$$

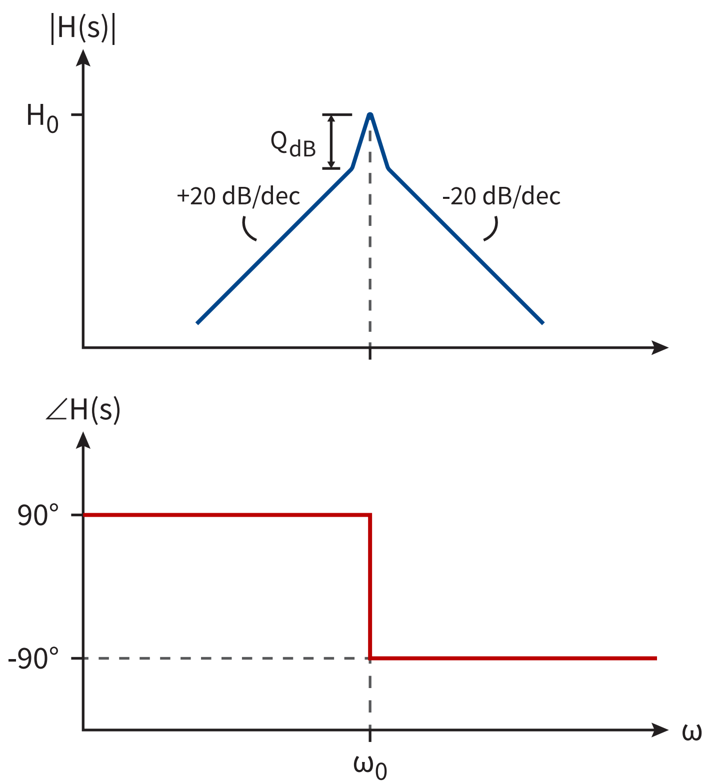


Figure 8: Sallen-Key band-pass filter Bode plot.

### 3.1 Sensitivity

$$S_{R_1}^{H_0} = \frac{H_0}{K} - 1 \tag{59}$$

$$S_{R_2}^{H_0} = \frac{H_0}{K} \cdot \frac{R_1}{R_2} \left(1 + \frac{C_1}{C_2}\right) \tag{60}$$

$$S_{R_3}^{H_0} = \frac{H_0}{K} \cdot \frac{R_1}{R_3} (1 - K) \tag{61}$$

$$S_{C_2}^{H_0} = -S_{C_1}^{H_0} = \frac{H_0}{K} \cdot \frac{R_1 C_1}{R_2 C_2} \tag{62}$$

$$S_{R_\mathrm{B}}^{H_0} = -S_{R_\mathrm{A}}^{H_0} = \left(H_0 \frac{R_1}{R_3} + 1\right) \frac{K-1}{K} \tag{63}$$

$$S_{R_1}^{\omega_0} = -\frac{R_3}{2 (R_1 + R_3)} \tag{64}$$

$$S_{R_2}^{\omega_0} = S_{C_1}^{\omega_0} = S_{C_2}^{\omega_0} = -\frac{1}{2} \tag{65}$$

$$S_{R_3}^{\omega_0} = -\frac{R_1}{2 (R_1 + R_3)} \tag{66}$$

$$S_{R_\mathrm{B}}^{\omega_0} = S_{R_\mathrm{A}}^{\omega_0} = 0 \tag{67}$$

$$S_{R_1}^{Q} = S_{R_1}^{H_0} + S_{R_1}^{\omega_0} + 1 \tag{68}$$

$$S_{R_2}^{Q} = S_{R_2}^{H_0} + S_{R_2}^{\omega_0} \tag{69}$$

$$S_{R_3}^{Q} = S_{R_3}^{H_0} + S_{R_3}^{\omega_0} \tag{70}$$

$$S_{C_2}^{Q} = -S_{C_1}^{Q} = S_{C_2}^{H_0} + S_{C_2}^{\omega_0} \tag{71}$$

$$S_{R_\mathrm{B}}^{Q} = -S_{R_\mathrm{A}}^{Q} = S_{R_\mathrm{B}}^{H_0} + S_{R_\mathrm{B}}^{\omega_0} - \frac{K-1}{K} \tag{72}$$

### 3.2 Components as Ratios

$$\begin{aligned} R_1 &= mR \\ R_2 &= nR \\ R_3 &= R \\ C_1 &= oC \\ C_2 &= C \end{aligned} \tag{73}$$

$$H_0 = \frac{K}{1 + (1-K)\,m + (1+o)\,m/n} \tag{74}$$

$$\omega_0 = \frac{\sqrt{1+m}}{RC\sqrt{mno}} \tag{75}$$

$$Q = \frac{\sqrt{1+m}\cdot\sqrt{mno}}{[1 + (1-K)\,m]\,n + m\,(1+o)} \tag{76}$$

### 3.2.1 Minimizing Resistor and Capacitor Sensitivities

$$S^Q_{R_1} = \frac{1}{1 + (1-K)\,m + (1+o)\,m/n} - \frac{1}{2\,(1+m)} \tag{77}$$

$$S^Q_{R_2} = \frac{(1+o)\,m/n}{1 + (1-K)\,m + (1+o)\,m/n} - \frac{1}{2} \tag{78}$$

$$S^Q_{R_3} = \frac{(1-K)\,m}{1 + (1-K)\,m + (1+o)\,m/n} - \frac{m}{2\,(1+m)} \tag{79}$$

$$S^Q_{C_2} = -S^Q_{C_1} = \frac{mo/n}{1 + (1-K)\,m + (1+o)\,m/n} - \frac{1}{2} \tag{80}$$

Minimizing the component sensitivities is more easily solved numerically with constraints set on the upper and lower bounds of the component ratios.

If $C$ is chosen, then $R$ can be calculated from $\omega_0$ as

$$R = \frac{\sqrt{1+m}}{\omega_0 C\sqrt{mno}} \tag{81}$$

### 3.2.2 Example MATLAB Script

```
%% Specifications

K = 1;
H0 = 0.5;
w0 = 2*pi*1e3;
Q = 1.5;
C = 10e-9;

```

```
%% Sensitivity optimization

x0 = ones(1, 3); % Initial values for m, n, and o
lb = 0.01 * ones(1, length(x0)); % Lower bound
ub = 100 * ones(1, length(x0)); % Upper bound

fun_wrapper = @(x) fun(x, K);
nonlcon_wrapper = @(x) nonlcon(x, K, H0, Q);
[x, fval] = fmincon(fun_wrapper, x0, [], [], [], [],
    lb, ub, nonlcon_wrapper);

m = x(1);
n = x(2);
o = x(3);
R = sqrt(1 + m)/(w0*C*sqrt(m*n*o));

%% Functions

function f = fun(x, K)
    m = x(1);
    n = x(2);
    o = x(3);

    H0_K = 1/(1 + (1 - K)*m + (1 + o)*m/n);
    S_Q_R1 = H0_K - 1/(2*(1 + m));
    S_Q_R2 = H0_K*((1 + o)*m/n) - 1/2;
    S_Q_R3 = H0_K*((1 - K)*m) - m/(2*(1 + m));
    S_Q_C2 = H0_K*(m*o/n) - 1/2;
    S_Q_C1 = -S_Q_C2;

    % Sum of all the absolute sensitivities to be
       minimized
    f = abs(S_Q_R1) + abs(S_Q_R2) + abs(S_Q_R3) + abs
       (S_Q_C1) + abs(S_Q_C2);
end

function [c, ceq] = nonlcon(x, K, H0, Q)
    m = x(1);
    n = x(2);
    o = x(3);
```

```
    c = [];

    % Constrain the parameters to meet the specified
       H0 and Q
    ceq(1) = H0 - K/(1 + (1 - K)*m + (1 + o)*m/n);
    ceq(2) = Q - (sqrt(1 + m)*sqrt(m*n*o))/((1 + (1 -
        K)*m)*n + (1 + o)*m);
end
```

### 3.2.3 Monte Carlo Simulation

The circuit used to generate the plot was designed for $K = 1$, $H_0 = 0.5$, $\omega_0 = 2\pi \cdot 1000$, and $Q = 1.5$. The resistor and capacitor tolerances were set to 1% and 5%, respectively.

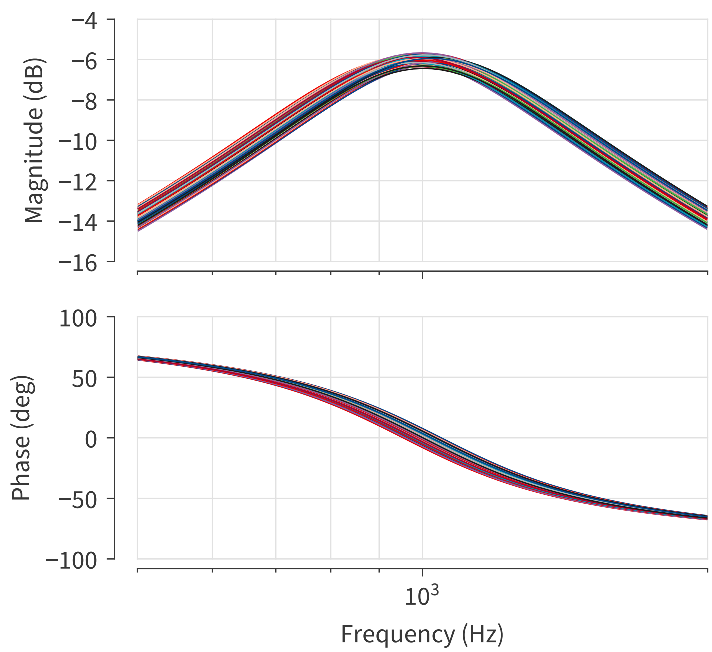


Figure 9: Sallen-Key band-pass filter Bode plot of 100 Monte Carlo samples.

### 3.3 Equal Capacitors and K = 1

$$\begin{aligned} C_1 = C_2 &= C \\ K &= 1 \end{aligned} \tag{82}$$

$$H_0 = \frac{1}{1 + 2R_1/R_2} \tag{83}$$

$$\omega_0 = \frac{\sqrt{1 + R_1/R_3}}{C\sqrt{R_1 R_2}} \tag{84}$$

$$Q = \frac{\sqrt{1 + R_1/R_3} \cdot \sqrt{R_1 R_2}}{2R_1 + R_2} \tag{85}$$

If $C$ is chosen, then

$$R_1 = \frac{Q}{H_0 \omega_0 C} \tag{86}$$

$$R_2 = \frac{2Q}{(1 - H_0)\, \omega_0 C} \tag{87}$$

$$R_3 = \frac{(1 - H_0)\, Q}{\left(H_0^2 - H_0 + 2Q^2\right) \omega_0 C} \tag{88}$$

### 3.4 Equal Components

$$\begin{aligned} R_1 = R_2 = R_3 &= R \\ C_1 = C_2 &= C \end{aligned} \tag{89}$$

$$H_0 = \frac{K}{4 - K} \tag{90}$$

$$\omega_0 = \frac{\sqrt{2}}{RC} \tag{91}$$

$$Q = \frac{\sqrt{2}}{4 - K} \tag{92}$$

## References


[1] S. Franco, *Design with Operational Amplifiers and Analog Integrated Circuits*, 4th ed. McGraw Hill, 2015, ISBN: 978-0-07-802816-8.

[2] L. P. Huelsman and P. E. Allen, *Introduction to the Theory and Design of Active Filters*. McGraw Hill, 1980, ISBN: 978-0-07-030854-1.

[3] Maxim Integrated, "Minimizing Component-Variation Sensitivity in Single Op Amp Filters," Application Note 738, Jul. 22, 2002.

[4] Texas Instruments, "OA-28 Low-Sensitivity, Bandpass Filter Design With Tuning Method," Application Note SNOA373C, Apr. 2013.