Op-Amp Sallen-Key Filter Design

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The relative sensitivity of y to x is defined as

$$S_x^y = \frac{\partial y / y}{\partial x / x} = \left(\frac{\partial y}{\partial x} \right) \frac{x}{y} \tag{1}$$

and is useful for optimizing component values in filter designs. Designing for low sensitivity is a way to minimize variations in filter parameters due to component tolerances.

1 Low-Pass Filter

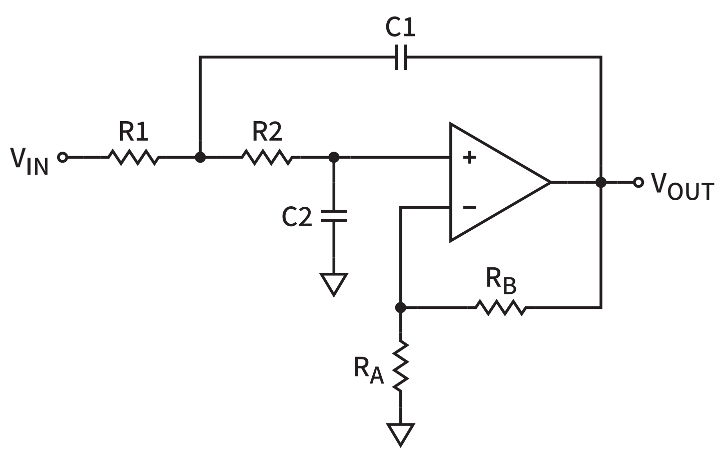

Figure 1: Sallen-Key low-pass filter schematic.

$$H(s) = H_0 \frac{1}{s^2/{\omega_0}^2 + s/(\omega_0 Q) + 1} \tag{2}$$

$$H_0 = K = 1 + R_\mathrm{B}/R_\mathrm{A} \tag{3}$$

$$\omega_0 = \frac{1}{\sqrt{R_1 R_2 C_1 C_2}} \tag{4}$$

$$Q = \frac{\sqrt{R_1 R_2 C_1 C_2}}{(1-K)\, R_1 C_1 + (R_1 + R_2)\, C_2} \tag{5}$$

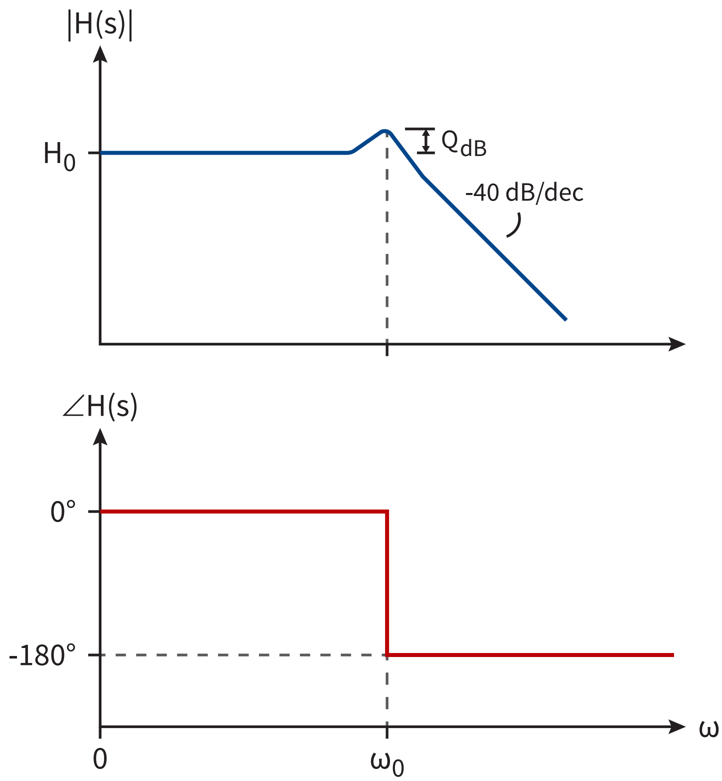

Figure 2: Sallen-Key low-pass filter Bode plot.

1.1 Sensitivity

$$S_{R_1}^{\omega_0} = S_{R_2}^{\omega_0} = S_{C_1}^{\omega_0} = S_{C_2}^{\omega_0} = -\frac{1}{2} \tag{6}$$

$$S_{R_B}^{\omega_0} = S_{R_A}^{\omega_0} = 0 \tag{7}$$

$$S_{R_1}^{Q} = -S_{R_2}^{Q} = \omega_0 Q R_2 C_2 - \frac{1}{2} \tag{8}$$

$$S_{C_1}^{Q} = -S_{C_2}^{Q} = \omega_0 Q \left(R_1 C_2 + R_2 C_2\right) - \frac{1}{2} \tag{9}$$

$$S_{R_B}^{Q} = -S_{R_A}^{Q} = \omega_0 Q \left(K - 1\right) R_1 C_1 \tag{10}$$

1.2 Components as Ratios

$$\begin{aligned} R_1 &= mR \\ R_2 &= R \\ C_1 &= nC \\ C_2 &= C \end{aligned} \tag{11}$$

$$\omega_0 = \frac{1}{RC\sqrt{mn}} \tag{12}$$

$$Q = \frac{\sqrt{mn}}{\left(1 - K\right) mn + m + 1} \tag{13}$$

1.2.1 Minimizing Resistor Sensitivity

$S_{R_1}^{Q} = S_{R_2}^{Q} = 0$ for

$$\omega_0 Q R_2 C_2 = \frac{1}{\left(1 - K\right) mn + m + 1} = \frac{1}{2} \tag{14}$$

$$\implies n = \frac{m - 1}{\left(K - 1\right) m} \tag{15}$$

Substituting n into Q and solving for m,

$$m = 1 + 4Q^2 \left(K - 1\right) \tag{16}$$

$$\implies n = \frac{4Q^2}{1 + 4Q^2 \left(K - 1\right)} \tag{17}$$

If C is chosen, then R can be calculated from ω_0 as

$$R = \frac{1}{2\omega_0 QC} \tag{18}$$

1.2.2 Minimizing Capacitor Sensitivity

$S_{C_1}^{Q} = S_{C_2}^{Q} = 0$ for

$$\omega_0 Q \left(R_1 C_2 + R_2 C_2\right) = \frac{m+1}{(1-K)\, mn + m + 1} = \frac{1}{2} \tag{19}$$

However, the ratio expression has a minimum value of 1, so the minimum capacitor sensitivity is $S_{C_1}^{Q} = -S_{C_2}^{Q} = 1/2$ and occurs when $K = 1$.

1.2.3 Monte Carlo Simulation

The circuit used to generate the plot was designed for $K = 1$, $\omega_0 = 2\pi \cdot 1000$, and $Q = 1.5$. The resistor and capacitor tolerances were set to 1% and 5%, respectively.

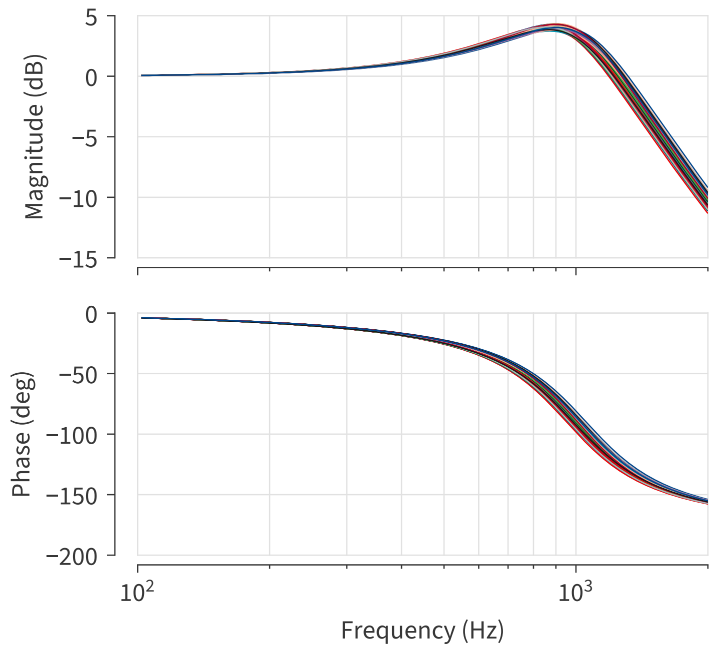

Figure 3: Sallen-Key low-pass filter Bode plot of 100 Monte Carlo samples.

1.3 Resistors as Ratios and Equal Capacitors

$$\begin{aligned} R_1 &= mR \\ R_2 &= R \\ C_1 = C_2 &= C \end{aligned} \tag{20}$$

$$\omega_0 = \frac{1}{RC\sqrt{m}} \tag{21}$$

$$Q = \frac{\sqrt{m}}{(2-K)\,m+1} \tag{22}$$

If C is chosen, then

$$m = \frac{4Q^2}{\left[1 + \sqrt{1 + 4Q^2 (K-2)}\right]^2} \tag{23}$$

$$R = \frac{1 + \sqrt{1 + 4Q^2 (K-2)}}{2\omega_0 QC} \tag{24}$$

1.4 Equal Components

$$\begin{aligned} R_1 &= R_2 = R \\ C_1 &= C_2 = C \end{aligned} \tag{25}$$

$$\omega_0 = \frac{1}{RC} \tag{26}$$

$$Q = \frac{1}{3-K} \tag{27}$$

2 High-Pass Filter

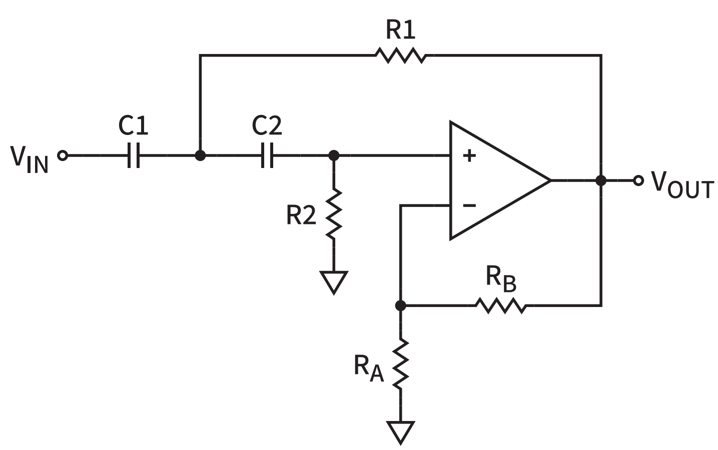

Figure 4: Sallen-Key high-pass filter schematic.

$$
\begin{aligned}
H(s) &= H_0 \frac{s^2/{\omega_0}^2}{s^2/{\omega_0}^2 + s/(\omega_0 Q) + 1} && (28)\\
H_0 &= K = 1 + R_\mathrm{B}/R_\mathrm{A} && (29)\\
\omega_0 &= \frac{1}{\sqrt{R_1 R_2 C_1 C_2}} && (30)\\
Q &= \frac{\sqrt{R_1 R_2 C_1 C_2}}{(1-K)\, R_2 C_2 + R_1\,(C_1 + C_2)} && (31)
\end{aligned}
$$

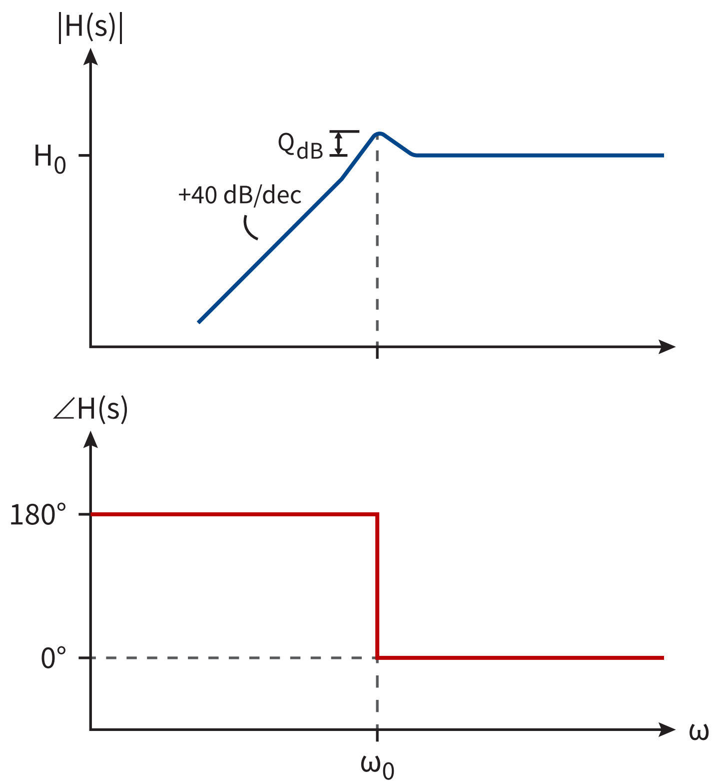

Figure 5: Sallen-Key high-pass filter Bode plot.

2.1 Sensitivity

$$S_{R_1}^{\omega_0} = S_{R_2}^{\omega_0} = S_{C_1}^{\omega_0} = S_{C_2}^{\omega_0} = -\frac{1}{2} \tag{32}$$

$$S_{R_B}^{\omega_0} = S_{R_A}^{\omega_0} = 0 \tag{33}$$

$$S_{R_2}^{Q} = -S_{R_1}^{Q} = \omega_0 Q \left(R_1 C_1 + R_1 C_2\right) - \frac{1}{2} \tag{34}$$

$$S_{C_2}^{Q} = -S_{C_1}^{Q} = \omega_0 Q R_1 C_1 - \frac{1}{2} \tag{35}$$

$$S_{R_B}^{Q} = -S_{R_A}^{Q} = \omega_0 Q \left(K - 1\right) R_2 C_2 \tag{36}$$

2.2 Components as Ratios

$$\begin{aligned} R_1 &= mR \\ R_2 &= R \\ C_1 &= nC \\ C_2 &= C \end{aligned} \tag{37}$$

$$\omega_0 = \frac{1}{RC\sqrt{mn}} \tag{38}$$

$$Q = \frac{\sqrt{mn}}{(1-K) + m\left(1+n\right)} \tag{39}$$

2.2.1 Minimizing Capacitor Sensitivity

$S_{C_2}^{Q} = S_{C_1}^{Q} = 0$ for

$$\omega_0 Q R_1 C_1 = \frac{mn}{(1-K) + m\left(1+n\right)} = \frac{1}{2} \tag{40}$$

$$\implies n = \frac{(1-K) + m}{m} \tag{41}$$

Substituting n into Q and solving for m,

$$m = \frac{1}{4Q^2} + \left(K - 1\right) \tag{42}$$

$$\implies n = \frac{1}{1 + 4Q^2\left(K-1\right)} \tag{43}$$

If C is chosen, then R can be calculated from ω_0 as

$$R = \frac{2Q}{\omega_0 C} \tag{44}$$

2.2.2 Minimizing Resistor Sensitivity

$S_{R_2}^{Q} = S_{R_1}^{Q} = 0$ for

$$\omega_0 Q \left(R_1 C_1 + R_1 C_2\right) = \frac{m\left(1+n\right)}{\left(1-K\right) + m\left(1+n\right)} = \frac{1}{2} \tag{45}$$

However, the ratio expression has a minimum value of 1, so the minimum resistor sensitivity is $S_{R_2}^{Q} = -S_{R_1}^{Q} = 1/2$ and occurs when $K = 1$.

2.2.3 Monte Carlo Simulation

The circuit used to generate the plot was designed for $K = 1$, $\omega_0 = 2\pi \cdot 1000$, and $Q = 1.5$. The resistor and capacitor tolerances were set to 1% and 5%, respectively.

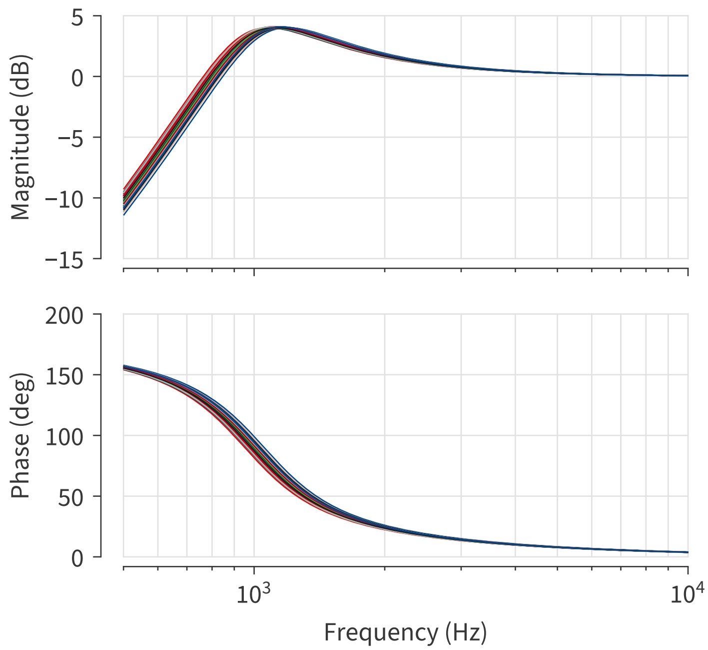

Figure 6: Sallen-Key high-pass filter Bode plot of 100 Monte Carlo samples.

2.3 Resistors as Ratios and Equal Capacitors

$$\begin{aligned} R_1 &= mR \\ R_2 &= R \\ C_1 = C_2 &= C \end{aligned} \tag{46}$$

$$\omega_0 = \frac{1}{RC\sqrt{m}} \tag{47}$$

$$Q = \frac{\sqrt{m}}{(1-K)+2m} \tag{48}$$

If C is chosen, then

$$m = \frac{\left[1 + \sqrt{1 + 8Q^2 (K-1)}\right]^2}{16Q^2} \tag{49}$$

$$R = \frac{4Q}{\left[1 + \sqrt{1 + 8Q^2 (K-1)}\right] \omega_0 C} \tag{50}$$

2.4 Equal Components

$$\begin{aligned} R_1 &= R_2 = R \\ C_1 &= C_2 = C \end{aligned} \tag{51}$$

$$\omega_0 = \frac{1}{RC} \tag{52}$$

$$Q = \frac{1}{3-K} \tag{53}$$

3 Band-Pass Filter

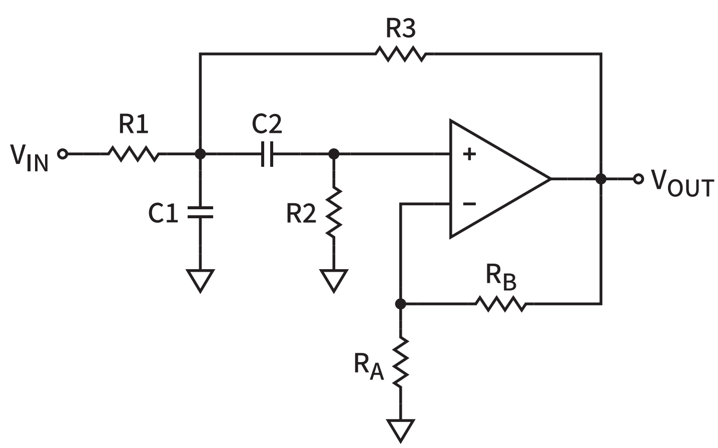

Figure 7: Sallen-Key band-pass filter schematic.

$$H(s) = H_0 \frac{s/(\omega_0 Q)}{s^2/{\omega_0}^2 + s/(\omega_0 Q) + 1} \tag{54}$$

$$K = 1 + R_B/R_A \tag{55}$$

$$H_0 = \frac{K}{1 + (1-K)\, R_1/R_3 + (R_1/R_2)\,(1 + C_1/C_2)} \tag{56}$$

$$\omega_0 = \frac{\sqrt{1 + R_1/R_3}}{\sqrt{R_1 R_2 C_1 C_2}} \tag{57}$$

$$Q = \frac{\sqrt{1 + R_1/R_3} \cdot \sqrt{R_1 R_2 C_1 C_2}}{[1 + (1-K)\, R_1/R_3]\, R_2 C_2 + R_1\,(C_1 + C_2)} \tag{58}$$

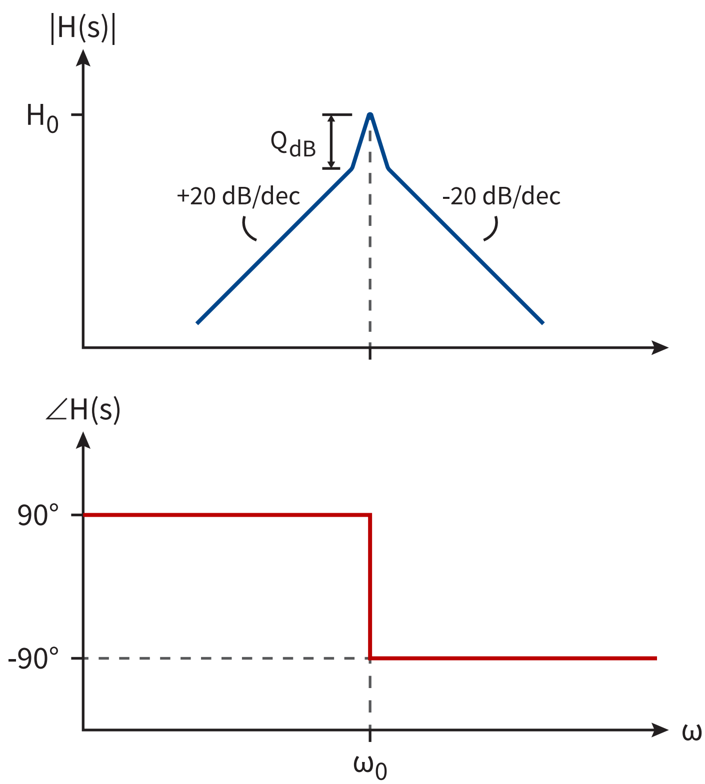

Figure 8: Sallen-Key band-pass filter Bode plot.

3.1 Sensitivity

$$S_{R_1}^{H_0} = \frac{H_0}{K} - 1 \tag{59}$$

$$S_{R_2}^{H_0} = \frac{H_0}{K} \cdot \frac{R_1}{R_2}\left(1 + \frac{C_1}{C_2}\right) \tag{60}$$

$$S_{R_3}^{H_0} = \frac{H_0}{K} \cdot \frac{R_1}{R_3}\left(1 - K\right) \tag{61}$$

$$S_{C_2}^{H_0} = -S_{C_1}^{H_0} = \frac{H_0}{K} \cdot \frac{R_1 C_1}{R_2 C_2} \tag{62}$$

$$S_{R_B}^{H_0} = -S_{R_A}^{H_0} = \left(H_0 \frac{R_1}{R_3} + 1\right)\frac{K-1}{K} \tag{63}$$

$$S_{R_1}^{\omega_0} = -\frac{R_3}{2\left(R_1 + R_3\right)} \tag{64}$$

$$S_{R_2}^{\omega_0} = S_{C_1}^{\omega_0} = S_{C_2}^{\omega_0} = -\frac{1}{2} \tag{65}$$

$$S_{R_3}^{\omega_0} = -\frac{R_1}{2\left(R_1 + R_3\right)} \tag{66}$$

$$S_{R_B}^{\omega_0} = S_{R_A}^{\omega_0} = 0 \tag{67}$$

$$S_{R_1}^{Q} = S_{R_1}^{H_0} + S_{R_1}^{\omega_0} + 1 \tag{68}$$

$$S_{R_2}^{Q} = S_{R_2}^{H_0} + S_{R_2}^{\omega_0} \tag{69}$$

$$S_{R_3}^{Q} = S_{R_3}^{H_0} + S_{R_3}^{\omega_0} \tag{70}$$

$$S_{C_2}^{Q} = -S_{C_1}^{Q} = S_{C_2}^{H_0} + S_{C_2}^{\omega_0} \tag{71}$$

$$S_{R_B}^{Q} = -S_{R_A}^{Q} = S_{R_B}^{H_0} + S_{R_B}^{\omega_0} - \frac{K-1}{K} \tag{72}$$

3.2 Components as Ratios

$$\begin{aligned} R_1 &= mR \\ R_2 &= nR \\ R_3 &= R \\ C_1 &= oC \\ C_2 &= C \end{aligned} \tag{73}$$

$$H_0 = \frac{K}{1 + (1 - K)\, m + (1 + o)\, m/n} \tag{74}$$

$$\omega_0 = \frac{\sqrt{1+m}}{RC\sqrt{mno}} \tag{75}$$

$$Q = \frac{\sqrt{1+m} \cdot \sqrt{mno}}{[1 + (1 - K)\, m]\, n + m\, (1 + o)} \tag{76}$$

3.2.1 Minimizing Resistor and Capacitor Sensitivities

$$S^Q_{R_1} = \frac{1}{1 + (1 - K)\, m + (1 + o)\, m/n} - \frac{1}{2\,(1+m)} \tag{77}$$

$$S^Q_{R_2} = \frac{(1 + o)\, m/n}{1 + (1 - K)\, m + (1 + o)\, m/n} - \frac{1}{2} \tag{78}$$

$$S^Q_{R_3} = \frac{(1 - K)\, m}{1 + (1 - K)\, m + (1 + o)\, m/n} - \frac{m}{2\,(1+m)} \tag{79}$$

$$S^Q_{C_2} = -S^Q_{C_1} = \frac{mo/n}{1 + (1 - K)\, m + (1 + o)\, m/n} - \frac{1}{2} \tag{80}$$

Minimizing the component sensitivities is more easily solved numerically with constraints set on the upper and lower bounds of the component ratios.

If C is chosen, then R can be calculated from ω_0 as

$$R = \frac{\sqrt{1+m}}{\omega_0 C\sqrt{mno}} \tag{81}$$

3.2.2 Example MATLAB Script

```
%% Specifications

K = 1;
H0 = 0.5;
w0 = 2*pi*1e3;
Q = 1.5;
C = 10e-9;

```

```
%% Sensitivity optimization

x0 = ones(1, 3); % Initial values for m, n, and o
lb = 0.01 * ones(1, length(x0)); % Lower bound
ub = 100 * ones(1, length(x0)); % Upper bound

fun_wrapper = @(x) fun(x, K);
nonlcon_wrapper = @(x) nonlcon(x, K, H0, Q);
[x, fval] = fmincon(fun_wrapper, x0, [], [], [], [], lb, ub, nonlcon_wrapper);

m = x(1);
n = x(2);
o = x(3);
R = sqrt(1 + m)/(w0*C*sqrt(m*n*o));

%% Functions

function f = fun(x, K)
    m = x(1);
    n = x(2);
    o = x(3);

    H0_K = 1/(1 + (1 - K)*m + (1 + o)*m/n);
    S_Q_R1 = H0_K - 1/(2*(1 + m));
    S_Q_R2 = H0_K*((1 + o)*m/n) - 1/2;
    S_Q_R3 = H0_K*((1 - K)*m) - m/(2*(1 + m));
    S_Q_C2 = H0_K*(m*o/n) - 1/2;
    S_Q_C1 = -S_Q_C2;

    % Sum of all the absolute sensitivities to be minimized
    f = abs(S_Q_R1) + abs(S_Q_R2) + abs(S_Q_R3) + abs(S_Q_C1) + abs(S_Q_C2);
end

function [c, ceq] = nonlcon(x, K, H0, Q)
    m = x(1);
    n = x(2);
    o = x(3);
```

```
    c = [];

    % Constrain the parameters to meet the specified H0 and Q
    ceq(1) = H0 - K/(1 + (1 - K)*m + (1 + o)*m/n);
    ceq(2) = Q - (sqrt(1 + m)*sqrt(m*n*o))/((1 + (1 - K)*m)*n + (1 + o)*m);
end
```

3.2.3 Monte Carlo Simulation

The circuit used to generate the plot was designed for $K = 1$, $H_0 = 0.5$, $\omega_0 = 2\pi \cdot 1000$, and $Q = 1.5$. The resistor and capacitor tolerances were set to 1% and 5%, respectively.

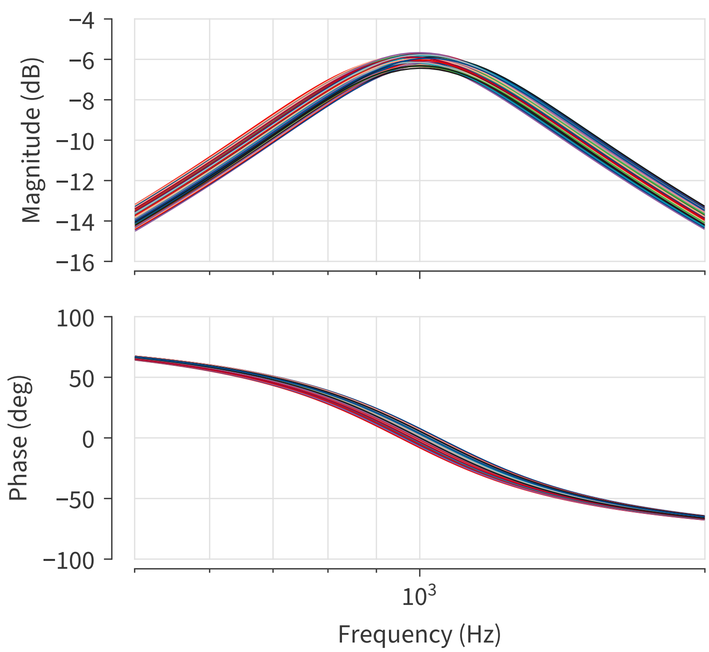

Figure 9: Sallen-Key band-pass filter Bode plot of 100 Monte Carlo samples.

3.3 Equal Capacitors and K = 1

$$
\begin{aligned}
C_1 = C_2 &= C \\
K &= 1
\end{aligned}
\tag{82}
$$

$$
H_0 = \frac{1}{1 + 2R_1/R_2} \tag{83}
$$

$$
\omega_0 = \frac{\sqrt{1 + R_1/R_3}}{C\sqrt{R_1 R_2}} \tag{84}
$$

$$
Q = \frac{\sqrt{1 + R_1/R_3} \cdot \sqrt{R_1 R_2}}{2R_1 + R_2} \tag{85}
$$

If C is chosen, then

$$
R_1 = \frac{Q}{H_0 \omega_0 C} \tag{86}
$$

$$
R_2 = \frac{2Q}{(1 - H_0)\, \omega_0 C} \tag{87}
$$

$$
R_3 = \frac{(1 - H_0)\, Q}{\left(H_0^2 - H_0 + 2Q^2\right) \omega_0 C} \tag{88}
$$

3.4 Equal Components

$$
\begin{aligned}
R_1 = R_2 = R_3 &= R \\
C_1 = C_2 &= C
\end{aligned}
\tag{89}
$$

$$
H_0 = \frac{K}{4 - K} \tag{90}
$$

$$
\omega_0 = \frac{\sqrt{2}}{RC} \tag{91}
$$

$$
Q = \frac{\sqrt{2}}{4 - K} \tag{92}
$$

References

[1] S. Franco, *Design with Operational Amplifiers and Analog Integrated Circuits*, 4th ed. McGraw Hill, 2015, ISBN: 978-0-07-802816-8.

[2] L. P. Huelsman and P. E. Allen, *Introduction to the Theory and Design of Active Filters*. McGraw Hill, 1980, ISBN: 978-0-07-030854-1.

[3] Maxim Integrated, "Minimizing Component-Variation Sensitivity in Single Op Amp Filters," Application Note 738, Jul. 22, 2002.

[4] Texas Instruments, "OA-28 Low-Sensitivity, Bandpass Filter Design With Tuning Method," Application Note SNOA373C, Apr. 2013.